MTH 512 - HW6

Fatimah Abdullah

-900085282-

32/32

Question(1).

Find M_T:

$$T(e_1) = T(1,0,0) = (1,2,4)$$

$$T(e_2) = T(0,1,0) = (1,0,0)$$

$$T(e_3) = T(0,0,1) = (0,3,6)$$

$$\therefore M_T = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 3 \\ 4 & 0 & 6 \end{bmatrix}$$

Define $T^*: \mathbb{R}^3 \to \mathbb{R}^3$.

$$M_{T^*} = (M_T)^T = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 3 & 6 \end{bmatrix}$$

$$T^*(v) = M_{T^*} V$$

Hence, $T^*(a,b,c) = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 3 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$= \begin{bmatrix} a + 2b + 4c \\ a \\ 3b + 6c \end{bmatrix}$$

$$\therefore T^*(a,b,c) = (a+2b+4c, a, 3b+6c).$$

Question(2).

Let $v \in M^{\perp}$. Then $\langle 1, v\rangle = 0$ and $\langle x^2, v\rangle = 0$

Since $v \in M^{\perp}$ and $M^{\perp}$ is a subspace of P_4,

then $v \in P_4$.

Hence, $v = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

Then we have:

$$\langle 1, v\rangle = \langle 1, a_0 + a_1 x + a_2 x^2 + a_3 x^3\rangle$$

$$= \int_0^1 (a_0 + a_1 x + a_2 x^2 + a_3 x^3)\, dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4} \right]_0^1$$

$$= a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = 0$$

$$\langle x^2, v\rangle = \langle x^2, a_0 + a_1 x + a_2 x^2 + a_3 x^3\rangle$$

$$= \int_0^1 (x^2)(a_0 + a_1 x + a_2 x^2 + a_3 x^3)\, dx$$

$$= \int_0^1 (a_0 x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5)\, dx$$

$$= \left[\frac{a_0 x^3}{3} + a_1 \frac{x^4}{4} + a_2 \frac{x^5}{5} + a_3 \frac{x^6}{6} \right]_0^1$$

$$= \frac{a_0}{3} + \frac{a_1}{4} + \frac{a_2}{5} + \frac{a_3}{6} = 0$$

Now, $M^{\perp}$ is the solution set of

$$\begin{cases} a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = 0 \\ \frac{a_0}{3} + \frac{a_1}{4} + \frac{a_2}{5} + \frac{a_3}{6} = 0 \end{cases}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & 0 \end{array}\right]$$

$$\xrightarrow{\text{REF}} \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 \\ 0 & 1 & \frac{16}{15} & 1 & 0 \end{array}\right]$$

$$a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \frac{1}{4}a_3 = 0 \rightarrow a_0 = -\frac{1}{2}a_1 - \frac{1}{3}a_2 - \frac{1}{4}a_3$$

$$a_1 + \frac{16}{15}a_2 + a_3 = 0 \rightarrow a_1 = -\frac{16}{15}a_2 - a_3$$

a_2 is a free variable

$$a_0 = -\frac{1}{2}\left(-\frac{16}{15}a_2 - a_3\right) - \frac{1}{3}a_2 - \frac{1}{4}a_3$$

$$a_0 = \frac{1}{5}a_2 + \frac{1}{4}a_3$$

a_3 is a free variable.

So, Sol. set $= \left\{ \left(\frac{1}{5}a_2 + \frac{1}{4}a_3, \ -\frac{16}{15}a_2 - a_3, \ a_2, a_3 \right) \middle| \ a_2, a_3 \in \mathbb{R} \right\}$

$= \left\{ a_2\left(\frac{1}{5}, \frac{-16}{15}, 1, 0\right) + a_3\left(\frac{1}{4}, -1, 0, 1\right) \middle| \ a_2, a_3 \in \mathbb{R} \right\}$

$= \text{span}\left\{ \left(\frac{1}{5}, \frac{-16}{15}, 1, 0\right), \left(\frac{1}{4}, -1, 0, 1\right) \right\}$

$\therefore M^{\perp} = \text{span}\left\{ \left(\frac{1}{5} - \frac{16}{15}x + x^2\right), \left(\frac{1}{4} - x + x^3\right) \right\}$.

Question(3).

$< f_1, f_2 > = \int_0^1 f_1 f_2 \, dx$ is an inner product on P_2.

And since $\mathbb{R}^2 \cong P_2$, then $\mathbb{R}^2 : (a,b) \rightarrow P_2 : (a+bx)$.

Hence: $\quad < (a_1+b_1x), (a_2+b_2x) > = \int_0^1 (a_1+b_1x)(a_2+b_2x) \, dx$

$$= \int_0^1 (a_1a_2 + a_1b_2x + a_2b_1x + b_1b_2x^2) \, dx$$

$$= \int_0^1 (a_1a_2 + (a_1b_2 + a_2b_1)x + b_1b_2x^2) \, dx$$

$$= \left[a_1a_2x + \frac{(a_1b_2 + a_2b_1)x^2}{2} + \frac{b_1b_2x^3}{3} \right]_0^1$$

$$= a_1a_2 + \frac{a_1b_2 + a_2b_1}{2} + \frac{b_1b_2}{3}$$

Since $< (a_1+b_1x), (a_2+b_2x) > = a_1a_2 + \frac{1}{2}(a_1b_2 + a_2b_1) + \frac{1}{3} b_1b_2$ is an inner product, then $< (a_1,b_1), (a_2,b_2) > = a_1a_2 + \frac{1}{2}(a_1b_2+a_2b_1) + \frac{1}{3} b_1b_2$ is also an inner product (due to the isomorphic).

Question (4).

Let $Q_1 = (a_1, a_2, a_3, a_4, a_5)$

$Q_2 = (b_1, b_2, b_3, b_4, b_5)$

Now, Cauchy-Schwarz inequality states that

$$\langle Q_1, Q_2 \rangle^2 \leq \langle Q_1, Q_1 \rangle \cdot \langle Q_2 \cdot Q_2 \rangle$$

$$\rightarrow \left[Q_1 \cdot Q_2^T\right]^2 \leq \left(Q_1 \cdot Q_1^T\right)\left(Q_2 \cdot Q_2^T\right)$$

$$\rightarrow \left(\begin{bmatrix} a_1, a_2, a_3, a_4, a_5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \right)^2 \leq \left(\begin{bmatrix} a_1, a_2, a_3, a_4, a_5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \right) \left(\begin{bmatrix} b_1, b_2, b_3, b_4, b_5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \right)$$

$$\rightarrow (a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5)^2 \leq (a_1a_1 + \cdots + a_5a_5)(b_1b_1 + \cdots + b_5b_5)$$

$$\therefore (a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5)^2 \leq (a_1^2 + \cdots + a_5^2)(b_1^2 + \cdots + b_5^2).$$

Question (5)

From Cauchy-Shwarz inequality, we Know that for $u, w \in V$,

we have $\langle u, w \rangle^2 \leq \langle u, u \rangle \langle w, w \rangle$

that is $\langle u, w \rangle^2 \leq \|u\|^2 \|w\|^2$

that is $|\langle u, w \rangle| \leq \|u\| \|w\|$

Now, we have :

$$
\begin{aligned}
\|V+W\|^2 &= \langle V+W, V+W \rangle \\
&= \langle V+W, V \rangle + \langle V+W, W \rangle \\
&= \langle V, V \rangle + \langle W, V \rangle + \langle V, W \rangle + \langle W, W \rangle \\
&= \langle V, V \rangle + \langle V, W \rangle + \langle V, W \rangle + \langle W, W \rangle \\
&= \langle V, V \rangle + 2\langle V, W \rangle + \langle W, W \rangle \\
&= \|V\|^2 + 2\langle V, W \rangle + \|W\|^2
\end{aligned}
$$

$\therefore \|V+W\|^2 = \|V\|^2 + 2\langle V, W \rangle + \|W\|^2$

since $|\langle u, w \rangle| \leq \|u\| \|w\|$, then

$\|V+W\|^2 \leq \|V\|^2 + 2\|u\| \|W\| + \|W\|^2$

$\|V+W\|^2 \leq (\|V\| + \|W\|)^2$

$\|V+W\| \leq \|V\| + \|W\|$. $\square$

Question (6)

Orthogonal Basis:

$$D = \text{span}\{\overset{Q_1}{1}, \overset{Q_2}{x^3}, \overset{Q_3}{x^4}\}$$

$$W_1 = Q_1 = 1$$

$$W_2 = Q_2 - \frac{\langle Q_2, W_1\rangle}{\|W_1\|^2} \cdot W_1 = Q_2 - \frac{\langle Q_2, W_1\rangle}{\langle W_1, W_1\rangle}$$

$$= x^3 - \frac{\langle x^3, 1\rangle}{\langle 1, 1\rangle} \cdot (1) = x^3 - \frac{\int_0^1 x^3\,dx}{\int_0^1 1\,dx} \cdot (1)$$

$$= x^3 - \frac{\left[\frac{x^4}{4}\right]_0^1}{[x]_0^1} \cdot (1) = x^3 - \frac{\frac{1}{4}}{1} \cdot (1) = x^3 - \frac{1}{4}.$$

$$W_3 = Q_3 - \frac{\langle Q_3, W_2\rangle}{\|W_2\|^2} \cdot W_2 - \frac{\langle Q_3, W_1\rangle}{\|W_1\|^2} \cdot W_1$$

$$= Q_3 - \frac{\langle Q_3, W_2\rangle}{\langle W_2, W_2\rangle} \cdot W_2 - \frac{\langle Q_3, W_1\rangle}{\langle W_1, W_1\rangle} \cdot W_1$$

$$= x^4 - \frac{\langle x^4, x^3 - \frac{1}{4}\rangle}{\langle x^3 - \frac{1}{4}, x^3 - \frac{1}{4}\rangle} \cdot \left(x^3 - \frac{1}{4}\right) - \frac{\langle x^4, 1\rangle}{\langle 1, 1\rangle} \cdot (1)$$

$$= x^4 - \frac{\int_0^1 x^4\left(x^3 - \frac{1}{4}\right)dx}{\int_0^1 \left(x^3 - \frac{1}{4}\right)^2 dx} \cdot \left(x^3 - \frac{1}{4}\right) - \frac{\int_0^1 x^4\,dx}{\int_0^1 1\,dx} \cdot (1)$$

$$= x^4 - \frac{\left[\frac{x^8}{8} - \frac{x^5}{20}\right]_0^1}{\left[\frac{x}{16} + \frac{x^7}{7} - \frac{x^4}{8}\right]_0^1} \cdot \left(x^3 - \frac{1}{4}\right) - \frac{\left[\frac{x^5}{5}\right]_0^1}{[x]_0^1} \cdot (1)$$

$$= x^4 - \frac{3/40}{9/112} \cdot \left(x^3 - \frac{1}{4}\right) - \frac{1/5}{1} \cdot (1)$$

$$= x^4 - \frac{14}{15}\left(x^3 - \frac{1}{4}\right) - \frac{1}{5}$$

$$= x^4 - \frac{14}{15}x^3 + \frac{14}{60} - \frac{1}{5}$$

$$= x^4 - \frac{14}{15}x^3 + \frac{1}{30}$$

Orthonormal Basis:

$$F_1 = \frac{w_1}{\|w_1\|} = \frac{w_1}{\sqrt{\langle w_1, w_1 \rangle}}$$

- $\|w_1\| = \sqrt{\langle w_1, w_1 \rangle} = \sqrt{1} = 1$

$$F_1 = \frac{1}{1} = 1$$

$$F_2 = \frac{w_2}{\|w_2\|} = \frac{w_2}{\sqrt{\langle w_2, w_2 \rangle}}$$

- $\|w_2\| = \sqrt{\langle w_2, w_2 \rangle} = \sqrt{9/112} = \frac{3\sqrt{7}}{28}$

$$F_2 = \frac{x^3 - \frac{1}{4}}{\frac{3\sqrt{7}}{28}} = \frac{4\sqrt{7}}{3}x^3 - \frac{\sqrt{7}}{3}$$

$$F_3 = \frac{w_3}{\|w_3\|} = \frac{w_3}{\sqrt{\langle w_3, w_3 \rangle}}$$

- $\|w_3\| = \sqrt{\langle w_3, w_3 \rangle} = \sqrt{1/900} = \frac{1}{30}$

$$F_3 = \frac{x^4 - \frac{14}{15}x^3 + \frac{1}{30}}{\frac{1}{30}} = 30x^4 - 28x^3 + 1$$

Question (7).

$$A = \begin{bmatrix} a & -3 \\ b & c \end{bmatrix}$$

Since A is positive definit, then $b = -3$,

and $ac - b^2 > 0$, then $ac - 9 > 0$

and we know $a, c > 0$

hence $ac > 9$

Question (8)

Solve $\langle 4x^3, 1-2x \rangle = \langle c_1(1-2x) + c_2 V_2 + c_3 V_3 + c_4 V_4 , 1-2x \rangle$

- $\langle 4x^3, 1-2x \rangle = \int_0^1 4x^3(1-2x)\,dx$

$$= \int_0^1 (4x^3 - 8x^4)\,dx$$

$$= \left[\frac{4x^4}{4} - \frac{8x^5}{5}\right]_0^1$$

$$= \left[x^4 - \frac{8}{5}x^5\right]_0^1$$

$$= 1 - \frac{8}{5}$$

$$= \frac{-3}{5}$$

- $\langle c_1(1-2x) + c_2V_2 + c_3V_3 + c_4V_4 \, , \, 1-2x \rangle$

$$= \langle c_1(1-2x), 1-2x \rangle + \langle c_2V_2, 1-2x \rangle + \langle c_3V_3, 1-2x \rangle + \langle c_4V_4, 1-2x \rangle$$

$$= c_1 \langle 1-2x, 1-2x \rangle + c_2 \cancel{\langle V_2, 1-2x \rangle} + c_3 \cancel{\langle V_3, 1-2x \rangle} + c_4 \cancel{\langle V_4, 1-2x \rangle}$$

Since $1-2x$ and V_2, V_3, V_4 are orthogonal, then their inner product is equal to zero.

$$= c_1 \langle 1-2x, 1-2x \rangle$$

$$= c_1 \int_0^1 (1-2x)^2 \, dx$$

$$= c_1 \int_0^1 (1-4x+4x^2) \, dx$$

$$= c_1 \left[x - \frac{4x^2}{2} + \frac{4x^3}{3} \right]_0^1$$

$$= c_1 \left[1 - \frac{4}{2} + \frac{4}{3} \right]$$

$$= \frac{1}{3} c_1$$

So, we have :

$$-\frac{3}{5} = \frac{1}{3} c_1$$

$$c_1 = \frac{-9}{5}$$